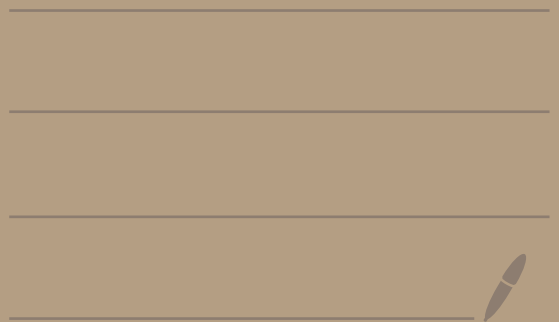


Math 4650
Homework 3
Solutions



①

$$\begin{aligned}\sum_{n=2}^{\infty} \left(\frac{2}{7}\right)^n &= \left(\frac{2}{7}\right)^2 + \left(\frac{2}{7}\right)^3 + \left(\frac{2}{7}\right)^4 + \dots \\ &= \left(\frac{2}{7}\right)^2 \left[1 + \left(\frac{2}{7}\right) + \left(\frac{2}{7}\right)^2 + \dots \right] \\ &\Rightarrow \left(\frac{2}{7}\right) \left[\frac{1}{1 - \frac{2}{7}} \right] = \frac{2}{7} \left[\frac{7}{5} \right] = \frac{2}{5}\end{aligned}$$

$$\begin{aligned}1 + x + x^2 + \dots &= \frac{1}{1-x} \\ \text{if } -1 < x < 1\end{aligned}$$

②

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{2n} &= \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^4 + \left(\frac{1}{3}\right)^6 + \left(\frac{1}{3}\right)^8 + \dots \\ &= \left(\frac{1}{3^2}\right) + \left(\frac{1}{3^2}\right)^2 + \left(\frac{1}{3^2}\right)^3 + \left(\frac{1}{3^2}\right)^4 + \dots \\ &= \frac{1}{9} + \left(\frac{1}{9}\right)^2 + \left(\frac{1}{9}\right)^3 + \left(\frac{1}{9}\right)^4 + \dots \\ &= \frac{1}{9} \left[1 + \frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \dots \right] \\ &= \frac{1}{9} \left[\frac{1}{1 - \frac{1}{9}} \right] = \frac{1}{9} \left[\frac{9}{8} \right] = \frac{1}{8}\end{aligned}$$

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad \text{if } -1 < x < 1$$

(3)

Use partial fractions:

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$$

$$1 = A(n+2) + B(n+1)$$

multiply by
 $(n+1)(n+2)$
on both sides

$$\underline{n=-2:} \quad 1 = A(-2+2) + B(-2+1)$$

$$1 = 0 - B$$

$$-1 = B$$

$$\underline{n=-1:} \quad 1 = A(-1+2) + B(-1+1)$$

$$1 = A(1) + 0$$

$$1 = A$$

$$\text{So, } \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

Let s_k be the partial sum sequence.

Then,

$$s_1 = \underbrace{\left(\frac{1}{2} - \frac{1}{3} \right)}_{n=1 \text{ term}}$$

$$S_2 = \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{n=1 \text{ term}} + \underbrace{\left(\frac{1}{3} - \frac{1}{4}\right)}_{n=2 \text{ term}} = \frac{1}{2} - \frac{1}{4}$$

$$S_3 = \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{n=1 \text{ term}} + \underbrace{\left(\frac{1}{3} - \frac{1}{4}\right)}_{n=2 \text{ term}} + \underbrace{\left(\frac{1}{4} - \frac{1}{5}\right)}_{n=3 \text{ term}} = \frac{1}{2} - \frac{1}{5}$$

•
•
•

In general,

$$S_k = \frac{1}{2} - \frac{1}{k+2}$$

Thus,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} &= \lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{k+2} \right) \\ &= \frac{1}{2} - 0 \\ &= \frac{1}{2} \end{aligned}$$

④

Use partial fractions:

$$\frac{2}{n^2-1} = \frac{2}{(n+1)(n-1)} = \frac{A}{n+1} + \frac{B}{n-1}$$

multiply
by
 $(n+1)(n-1)$
on both
sides

$$2 = A(n-1) + B(n+1)$$

$$\underline{n=1:} \quad 2 = A(1-1) + B(1+1)$$

$$2 = 0 + 2B$$

$$1 = B$$

$$\underline{n=-1:} \quad 2 = A(-1-1) + B(-1+1)$$

$$2 = A(-2) + 0$$

$$-1 = A$$

Thus,

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty} \left(\frac{-1}{n+1} + \frac{1}{n-1} \right)$$

Let s_k denote the partial sum sequence.

Then,

$$S_2 = \underbrace{\left(\frac{-1}{2+1} + \frac{1}{2-1} \right)}_{n=2 \text{ term}} = -\frac{1}{3} + 1 = \boxed{\frac{2}{3}}$$

$$S_3 = \underbrace{\left(-\frac{1}{3} + 1\right)}_{n=2 \text{ term}} + \underbrace{\left(-\frac{1}{4} + \frac{1}{2}\right)}_{n=3 \text{ term}} = \boxed{\frac{3}{2} - \frac{1}{3} - \frac{1}{4}}$$

$$S_4 = \underbrace{\left(-\cancel{\frac{1}{3}} + 1\right)}_{n=2 \text{ term}} + \underbrace{\left(-\frac{1}{4} + \frac{1}{2}\right)}_{n=3 \text{ term}} + \underbrace{\left(-\frac{1}{5} + \cancel{\frac{1}{3}}\right)}_{n=4 \text{ term}}$$

$$= 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} = \boxed{\frac{3}{2} - \frac{1}{4} - \frac{1}{5}}$$

$$S_5 = S_4 + \underbrace{\left(-\frac{1}{6} + \frac{1}{4}\right)}_{n=5 \text{ term}}$$

$$= \left(\frac{3}{2} - \cancel{\frac{1}{4}} - \frac{1}{5}\right) + \left(-\frac{1}{6} + \cancel{\frac{1}{4}}\right)$$

$$= \boxed{\frac{3}{2} - \frac{1}{5} - \frac{1}{6}}$$

$$S_6 = S_5 + \underbrace{\left(-\frac{1}{7} + \frac{1}{5}\right)}_{n=6 \text{ term}}$$

$$= \left(\frac{3}{2} - \cancel{\frac{1}{5}} - \frac{1}{6}\right) + \left(-\frac{1}{7} + \cancel{\frac{1}{5}}\right)$$

$$= \boxed{\frac{3}{2} - \frac{1}{6} - \frac{1}{7}}$$

⋮ ⋮

We see that in general

$$S_k = \frac{3}{2} - \frac{1}{k} - \frac{1}{k+1}$$

Thus,

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{2}{n^2-1} &= \lim_{k \rightarrow \infty} S_k \\ &= \lim_{k \rightarrow \infty} \left(\frac{3}{2} - \frac{1}{k} - \frac{1}{k+1} \right) \\ &= \frac{3}{2} - 0 - 0 \\ &= \frac{3}{2} \end{aligned}$$

⑤(a)

Suppose that $\sum_{n=1}^{\infty} a_n = A$ and $\alpha \in \mathbb{R}$.

Let $s_k = a_1 + a_2 + \dots + a_k$
be the partial sums for $\sum_{n=1}^{\infty} a_n$.

Since $\sum_{n=1}^{\infty} a_n = A$ we know $\lim_{k \rightarrow \infty} s_k = A$.

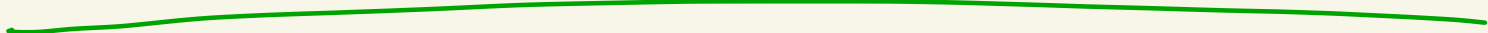
Note that the partial sums of $\sum_{n=1}^{\infty} \alpha a_n$ are

$$\alpha a_1 + \alpha a_2 + \dots + \alpha a_k = \alpha(a_1 + a_2 + \dots + a_k) \\ = \alpha s_k$$

Then,

$$\sum_{n=1}^{\infty} \alpha a_n = \lim_{k \rightarrow \infty} \alpha s_k = \alpha \lim_{k \rightarrow \infty} s_k = \alpha A.$$

algebra
of sequence
limits



⑤(b)

Suppose that $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$.

$$\text{Let } s_k = a_1 + a_2 + \dots + a_k$$

$$\text{and } t_k = b_1 + b_2 + \dots + b_k$$

be the partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$.

Then, $\lim_{k \rightarrow \infty} s_k = A$ and $\lim_{k \rightarrow \infty} t_k = B$.

The partial sums of $\sum_{n=1}^{\infty} (a_n + b_n)$ are

$$(a_1 + b_1) + (a_2 + b_2) + \dots + (a_k + b_k)$$

$$= (a_1 + a_2 + \dots + a_k) + (b_1 + b_2 + \dots + b_k)$$

$$= s_k + t_k$$

Thus, $\sum_{n=1}^{\infty} (a_n + b_n) = \lim_{k \rightarrow \infty} (s_k + t_k) = A + B$

↑
algebra of
sequence
limits



⑥

($\Rightarrow$) Suppose that (a_n) converges to 0.

Let $\varepsilon > 0$.

Then there exists $N > 0$ where if $n \geq N$
then $|a_n - 0| < \varepsilon$.

So if $n \geq N$ then $|a_n| < \varepsilon$

So if $n \geq N$ then $||a_n|| < \varepsilon$

So if $n \geq N$ then $||a_n| - 0| < \varepsilon$

Thus, $(|a_n|)$ converges to 0.

($\Leftarrow$) Suppose $(|a_n|)$ converges to 0.

Then there exists $N > 0$ where if $n \geq N$
then $||a_n| - 0| < \varepsilon$.

Thus if $n \geq N$ then $||a_n|| < \varepsilon$

So if $n \geq N$ then $|a_n| < \varepsilon$.

So if $n \geq N$ then $|a_n - 0| < \varepsilon$

Thus, (a_n) converges to 0.



⑦(a)

$$\lim_{n \rightarrow \infty} \frac{2n}{3n+1} = \lim_{n \rightarrow \infty} \frac{2}{3 + 1/n} = \frac{2}{3+0} = \frac{2}{3}$$

↑
divide top/bottom
by n

Since $\lim_{n \rightarrow \infty} \frac{2n}{3n+1} \neq 0$, by the divergence

test, we know $\sum_{n=1}^{\infty} \frac{2n}{3n+1}$ diverges.

⑦(b)

Note that $\frac{1}{n^3+n^2+n+1} \leq \frac{1}{n^3+n^3+n^3+n^3} = \frac{4}{n^3}$

↑
 $n \geq 1$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p=3 series)

Thus from HW problem 5 above $\sum_{n=1}^{\infty} \frac{4}{n^3}$ converges.

By the comparison test we know

that $\sum_{n=1}^{\infty} \frac{1}{n^3+n^2+n+1}$ converges

⑦(c)

Consider
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

where $a_n = \frac{1}{n^2+1}$ are positive real numbers.

We use the alternating series test to show this series converges.

We have that

$$a_{n+1} = \frac{1}{(n+1)^2+1} = \frac{1}{n^2+2n+2} < \frac{1}{n^2+1} = a_n$$

$$n^2+2n+2 > n^2+0+1 = n^2+1$$

So, the sequence (a_n) is monotonically decreasing.

Also,
$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n^2+1} = 0.$$

Thus, by the alternating series test

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1} \text{ converges.}$$

(7)(d)

$$\lim_{n \rightarrow \infty} \left| (-1)^n \frac{n^3}{2n^3 + 1} \right| = \lim_{n \rightarrow \infty} \underbrace{|(-1)^n|}_1 \cdot \left| \frac{n^3}{2n^3 + 1} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{n^3}{2n^3 + 1} = \lim_{n \rightarrow \infty} \frac{1}{2 + 1/n^3} = \frac{1}{2 + 0} = \frac{1}{2}$$

$$\frac{n^3}{2n^3 + 1} > 0$$

since $n \geq 1$

By problem 6 since $\lim_{n \rightarrow \infty} \left| (-1)^n \frac{n^3}{2n^3 + 1} \right| \neq 0$
We know $\lim_{n \rightarrow \infty} (-1)^n \frac{n^3}{2n^3 + 1} \neq 0$.

Thus, by the divergence test

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{2n^3 + 1} \text{ diverges.}$$

(7)(e) Let s_k be the partial sums of $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$. Then,

$$s_1 = \sqrt{2} - \sqrt{1} = \sqrt{2} - 1$$

$$s_2 = (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) = \sqrt{3} - 1$$

$$s_3 = s_2 + (\sqrt{4} - \sqrt{3}) = (\sqrt{3} - 1) + (\sqrt{4} - \sqrt{3}) = \sqrt{4} - 1$$

$\vdots$

$$\text{In general, } s_k = \sqrt{k} - 1$$

We see that (s_k) is unbounded so

$\lim_{k \rightarrow \infty} s_k$ does not exist.

Thus, $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$ diverges

Let $M > 0$.

Then, set $k = (M+2)^2$ to get $s_k = \sqrt{(M+2)^2} - 1 = M+1 > M$.

⑧ (a)

$$\text{Let } s_k = |a_1| + |a_2| + \dots + |a_k|$$

$$\text{and } t_k = a_1 + a_2 + \dots + a_k$$

be the partial sums of $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} a_n$

Since s_k converges, it is a Cauchy sequence

Let $\varepsilon > 0$.

Then there exists $N > 0$ where if $l \geq m \geq N$ then

$$|s_l - s_m| < \varepsilon.$$

That is if $l \geq m \geq N$, then

$$\begin{aligned} & |(|a_1| + |a_2| + \dots + |a_l|) - (|a_1| + |a_2| + \dots + |a_m|)| \\ &= | |a_{m+1}| + |a_{m+2}| + \dots + |a_l| | < \varepsilon \end{aligned}$$

So if $l \geq m \geq N$, then $|a_{m+1}| + |a_{m+2}| + \dots + |a_l| < \varepsilon$

Then if $l \geq m \geq N$, we have

$$\begin{aligned} |t_l - t_m| &= |a_{m+1} + a_{m+2} + \dots + a_l| \\ &\leq |a_{m+1}| + |a_{m+2}| + \dots + |a_l| < \varepsilon \end{aligned}$$

So, the sequence (t_k) is Cauchy also.

Thus, $\sum_{n=1}^{\infty} a_n$ converges.

⑧(b)

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges

but $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

} from
class

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If $-1 < r < 1$, then $|r| < 1$.

If $r = 0$, then $\lim_{n \rightarrow \infty} |r^n| = \lim_{n \rightarrow \infty} 0 = 0$

If $r \neq 0$ and $|r| < 1$, then $\lim_{n \rightarrow \infty} |r^n| = \lim_{n \rightarrow \infty} |r|^n = 0$
by HW 2.

Thus, $\lim_{n \rightarrow \infty} |r^n| = 0$

By problem 6, we get $\lim_{n \rightarrow \infty} r^n = 0$

